

# On the gauge invariant and topological nature of the localization determining the Quantum Hall Effect plateaus

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## Abstract

It is shown how the electromagnetic response of 2DEG under Quantum Hall Effect regime, characterized by the Chern-Simons topological action, transforms the sample impurities and defects in charge-reservoirs that stabilize the Hall conductivity plateaus. The results, determine the basic dynamical origin of the singular properties of localization under the occurrence of the Quantum Hall Effect obtained in the pioneering works of Laughlin and of Joynt and Prange, by means of a gauge invariance argument and a purely electronic analysis, respectively. The common intuitive picture of electrons moving along the equipotential lines gets an analytical realization through the Chern-Simons current and charge densities.

Since its discovery in 1980, the integer Quantum Hall Effect (QHE) has been the subject of a vast active research [1, 2]. It is widely accepted that the explanation of this rather remarkable effect, particularly the existence of Hall conductivity plateaus, is intimately connected with localization of electrons by sample impurities [2, 5, 6].

In the case of a perfect non-interacting two-dimensional electron gas (2DEG) at sufficiently low temperature and with an exactly integral number of Landau levels filled, the Hall conductivity  $\sigma_{yx} = n_e e c / B = n e^2 / h$ , where  $n$  is the integral number of filled Landau levels, and this is a consequence of the well known fact that each Landau level has a degeneracy  $eB/hc$  per unit area [7]. For the Hall conductivity to show these quantized values for wide ranges of the quantity  $n_e/B$ , which is indeed the essential feature of the QHE, there must be some sort of charge-reservoir in order to adjust the number of electrons in extended current-carrying states to that required by a quantized conductivity [8]. The charge-reservoir, it has been argued, is provided by sample impurities or imperfections, and the mechanism that of localization of electrons by the associated random potential. It has been shown that as long as the Fermi energy varies within a region of localized states the electrons in extended states carry the right current for the Hall conductivity to be quantized [5].

However, it is also recognized that a complete microscopic theory from which the properties of the effect could be deductively obtained, is still missing [3, 7]. The present work is intended to show how the electromagnetic response of the system, predicted by the field theory considered in [9, 10], can sustain a microscopic description from which some basic properties of the role of impurities for QHE could be derived. Here, it is evidenced that the electromagnetic response of the system, described by the Chern-Simons topological action, effectively transforms the sample impurities in charge-reservoirs. Moreover, for some ranges of variation of applied magnetic field, the system is capable of adjusting the electron density in most of the sample points to those values required for satisfying the integral filling of Landau levels locally. This picture gives, thus, a fundamental explanation of the results of references [4, 5], highlighting the gauge invariance and topological character of the effect. The results also lead ground to the applicability of the field description of integer QHE presented in [11], where the mechanism of chemical potential stabilization in a gap remained unclear.

To simplify the discussion the analysis has been carried on a multi quantum well structure

(superlattice) [12, 13], where the field distribution of interest can be obtained analytically. First, the field distribution associated to a model impurity in the superlattice is presented and its connection to the Chern-Simons topological action is shown. Next, it is discussed the stability of the QHE and the emergence of the Hall conductivity plateaus is seen.

The electromagnetic response  $a_\mu(x)$  represents a linear disturbance of the electromagnetic potential associated to the presence of the constant magnetic field  $B_n = n_e hc / ne$  at which the filling factor  $\nu$  has precisely the value  $n$ . In [9] it was found the equation satisfied by the electromagnetic response of a superlattice in QHE regime

$$\begin{aligned} \partial^2 a_\mu(x) + i \frac{4\pi\sigma_H}{ca} \epsilon^{\alpha\mu\sigma\nu} n_\alpha \partial_\sigma a_\nu(x) \\ + 4\pi \frac{\chi_e}{a} [P_{\mu\nu} u_\alpha u_\beta + u_\mu u_\nu P_{\alpha\beta} \\ - (u_\mu P_{\nu\alpha} + u_\nu P_{\mu\alpha}) u_\beta] \partial_\alpha \partial_\beta a_\nu(x) = 0, \end{aligned} \quad (1)$$

which, in order to evidence the topological Chern-Simons terms in it, has been written in Lorentz-covariant form. Also,  $a$  is the distance between the 2DEG planes in the superlattice,  $u_\mu = (1, 0, 0, 0)$  is the superlattice 4-velocity,  $n_\mu = (0, 0, 0, 1)$  is a unitary 4-vector normal to the 2DEG planes,  $P_{\mu\nu} = \text{diag}(0, 1, 1, 0)$  is the projection tensor on the 2DEG planes, and  $\sigma_H = ne^2/h$  and  $\chi_e = n \frac{e^2}{h} \frac{mc}{eB_n}$  are the Hall conductivity and the dielectric susceptibility of a single plane of electrons at filling factor  $\nu = n$ , respectively. The 4-potential  $a_\mu(x)$  is taken in Coulomb gauge and will be supposed independent of  $x_3$  coordinate. It will also be assumed that  $a_3 = 0$ .

The effective Maxwell equations (1) were obtained in [9] within the long wavelength approximation  $\lambda \gg r_0 = \sqrt{\frac{\hbar c}{eB_n}}$  from a calculation of the first quantum correction to the effective action of a 2DEG in the presence of a magnetic field. The equations appropriate to a layered multi quantum well structure are obtained by simply adding the equations for each plane. The above equations (1) also can be alternatively derived by extremizing with respect to  $a_\mu(x)$  the effective action

$$\Gamma_{\text{eff}}[a_\mu] = \int d^2x dt \left[ \frac{1}{8\pi} \epsilon E^2 - \frac{1}{8\pi} B^2 + \frac{i\sigma_H}{4ca} \epsilon^{\alpha\mu\nu} a_\alpha F_{\mu\nu} \right], \quad (2)$$

where  $\epsilon = 1 + 4\pi \frac{\chi_e}{a}$ ,  $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ ,  $E_j = -iF_{0j}$  and  $B = \frac{1}{2} \epsilon^{jk} F_{jk}$  ( $j, k = 1, 2$ ). The third term in the integrand of (2) is recognized as the Chern-Simons topological action, since is independent of the metric tensor.

In the present work, the impurity in superlattice sample will be modelled as a cylindrical hole of radius  $\eta$  having the axis normal to the 2DEG. Then, we search for stationary axially-symmetric solutions to (1), which in this case reduces to consider the following equations for the scalar potential  $\phi(r)$  and the azimuthal component of the vector potential  $a_\theta(r)$

$$\begin{aligned}\frac{d}{dr} \left( r \frac{d\phi}{dr} \right) &= -\frac{4\pi\sigma_H}{ca} \frac{1}{\epsilon} \frac{d}{dr} (r a_\theta), \\ \frac{1}{r} \frac{d}{dr} \left( r \frac{da_\theta}{dr} \right) - \frac{a_\theta}{r^2} &= \frac{4\pi\sigma_H}{ca} \frac{d\phi}{dr}.\end{aligned}$$

These equations were investigated in [14]. A solution, finite for  $r \rightarrow \infty$ , is given by

$$\begin{aligned}\phi(r) &= \frac{C}{\sqrt{\epsilon}} K_0(kr), \\ a_\theta(r) &= C \left( -\frac{1}{kr} + K_1(kr) \right),\end{aligned}\tag{3}$$

where  $C$  is an overall constant factor to be determined,  $K_0(x)$  and  $K_1(x)$  are MacDonald functions of 0th and 1st order respectively, and  $k = \frac{4\pi\sigma_H}{ca\sqrt{\epsilon}}$ . From these expressions, the response electromagnetic field of the superlattice is readily obtained as

$$E(r) = \frac{kC}{\sqrt{\epsilon}} K_1(kr), B(r) = -kC K_0(kr).\tag{4}$$

Note that the magnetic field is proportional to the electric potential. These fields decay exponentially for  $r \rightarrow \infty$ , being  $k^{-1}$  a measure of its effective extension, and will be adopted as valid for the region external to the cylinder  $r \geq \eta$ . Within the interior, for completely defining the impurity model, the electric potential will be assumed constant and the magnetic intensity will also be assumed to be a constant vector along the axis orthogonal to the superlattice planes.

Using the expression (4) for  $B(r)$ , the magnetic flux  $\Phi$  associated to the impurity can be calculated to be

$$\Phi = \int \vec{B} \cdot \vec{n} dS = \int_\eta^\infty B(r) 2\pi r dr = -2\pi C \eta K_1(k\eta),\tag{5}$$

which in the limit  $\eta \rightarrow 0$

$$\lim_{\eta \rightarrow 0} \Phi = -2\pi \frac{C}{k}.\tag{6}$$

Then, the edge current which flows through the boundary of the cylinder should be such that in conjunction with the continuous distributed Hall currents will produce the just

determined total magnetic flux. The explicit determination of this current is not necessary for the further steps of constructing the model.

Let us now consider the edge electric charge. Applying Gauss law to the cylinder, the free edge charge  $q$  per unit length in  $x_3$  direction in the impurity can be obtained as

$$q = \frac{1}{2}\eta\epsilon E(\eta^+) = \frac{1}{2}C\sqrt{\epsilon}k\eta K_1(k\eta), \quad (7)$$

and

$$\lim_{\eta \rightarrow 0} q = \frac{1}{2}C\sqrt{\epsilon}. \quad (8)$$

After substituting the solution (3) in the expression (2) for the effective action we get

$$\Gamma_{\text{eff}} = \frac{C^2}{4}K_0(k\eta) \int dt. \quad (9)$$

Using the fact that, for stationary equilibrium fields, the effective action is minus the space-time integral of the energy density, considering the contribution to the energy of the free edge charge, and using (3), (8) and (9), we obtain, for the mean energy density  $\mathcal{U}$  associated to an impurity, the expression

$$\mathcal{U} = -\frac{\Gamma_{\text{eff}}}{A \int dt} + \frac{q\phi(\eta^+)}{A} = \frac{C^2}{4A}K_0(k\eta), \quad (10)$$

where  $A$  is the sectional area of the sample. This is the energy required to produce a static deformation (3) of the integral filling homogeneous field  $B_n$ .

As the next step in the definition of the model, we will consider  $N$  impurities like that described above, spread over the sample area to a mean separation  $\xi \equiv \sqrt{A/N}$ . The holes will also be assumed to be small for having a total area very much smaller than the sample surface. Hence, the electromagnetic mean field associated to the inspected many electron state, should be approximately given by superposing the fields of all the impurities. An illustrative picture of the spacial dependence on the 2DEG planes of the electric potential is shown in Fig.1. At this point, it should be noticed that the field solving the Maxwell equations (1), exactly satisfy the following properties: a) The Hall current always flows along the equipotential level curves. b) The normal magnetic field to the planes and the charge densities at any point out of the cores of the impurities also exactly satisfies the integral filling condition, as it was discussed in [14]. Therefore, it can be concluded, that the

electric potential and the Hall currents of the considered many electron state, analytically realize the usual intuitive picture for the conduction in QHE and moreover, also gives ground for early proposed percolation models [2, 3].

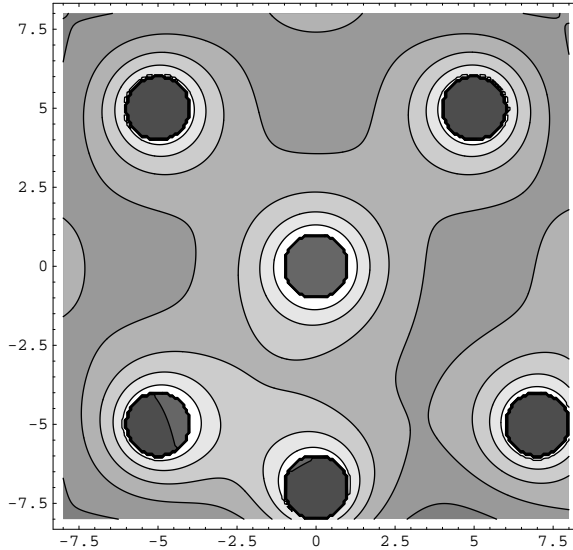


FIG. 1: Contour plot illustrating the equipotential curves of the superposed fields for various impurities. The Hall currents flow along them, and while some lines are closed, other ones connect different spatial regions reflecting the presence of extended states.

Let us resume the picture we intend to support in the argue to be done below. Whenever the external magnetic field  $B_{\text{ext}}$  slightly deviates from one of the values required by integral filling of Landau levels, each impurity accumulates a free edge charge  $q$  such that the electron density at any internal point in the sample volume, adjusts to that required to satisfy the integral filling condition at the precise local magnetic field value. For the sample type we have been considering, the response electromagnetic field is exponentially damped away from the impurities. Therefore, the excess of magnetic flux over that corresponding to integral filling flux density  $B_n$  will tend to concentrate more around the impurities. If the distance between them turns to be greater than the penetration length, then as mentioned before, a sort of Meissner effect occurs. Then, the difference between the external flux and the one associated to a complete filling of some number of Landau levels will be expelled from the large volumes and concentrated around the defects [9]. The defect properties will resemble in such regimes the vortices in type II superconductors. This should not be necessarily the case. It also should be stressed that, although in the realistic samples, the defects can be

expected to be distributed in uncorrelated positions on each plane of the superlattice, it is natural to suppose that the magnetic flux lines of the real defects will follow trajectories that traverse the sample in  $x_3$  direction with some distortion respect (like the real Abrikosov vortices in non-ideal samples) to the perfect line configuration we are considering.

In order to justify the above picture, it should be shown that the system energy is lower than the value corresponding to the purely homogeneous field configuration. Let us consider this problem in what follows.

To start, let us define the connections between the main parameters characterizing the model. From the condition of equality of total magnetic flux in and out of the sample,

$$B_{\text{ext}}A = B_nA + N\Phi,$$

and from the expression (6), the constant  $C$  is determined to be

$$C = -\frac{B_{\text{ext}} - B_n}{2\pi}k\xi^2.$$

Substituting this result in expression (10) and considering there are  $N$  impurities, it is obtained the energy density in QHE regime

$$\mathcal{U}_{\text{QHE}}(B_{\text{ext}}) = \frac{(B_{\text{ext}} - B_n)^2}{16\pi^2}k^2\xi^2K_0(k\eta) \quad (11)$$

It must be remarked, that here, in order to simplify the discussion, for evaluating this energy, it was assumed that the field distributions of different impurities do not overlap very much. This supposition was employed in order to approximately calculate the energy as the simple sum of the contributions associated to each defect. A more precise procedure is however, possible for the search of more quantitative results.

On another hand, when the homogeneous field  $B_{\text{ext}}$  completely penetrates the sample, the total energy density required for such deformation over the integral filling field  $B_n$  is calculated to be

$$\mathcal{U}_{\text{homog}}(B_{\text{ext}}) = \frac{(B_{\text{ext}} - B_n)^2}{8\pi} + \mathcal{U}_{\text{Peierls}}(B_{\text{ext}}), \quad (12)$$

where  $\mathcal{U}_{\text{Peierls}}$  is a contribution to the energy due to Peierls and described in [10]; it is given by

$$\mathcal{U}_{\text{Peierls}}(B_{\text{ext}}) = \frac{n_e^2 h^2}{4\pi m a} \left\{ -\left[\frac{B_1}{B_{\text{ext}}}\right] \left(\left[\frac{B_1}{B_{\text{ext}}}\right] + 1\right) \left(\frac{B_{\text{ext}}}{B_1}\right)^2 + \left(2\left[\frac{B_1}{B_{\text{ext}}}\right] + 1\right) \frac{B_{\text{ext}}}{B_1} - 1 \right\}, \quad (13)$$

where  $[x]$  denotes the integer part of  $x$ .

For the QHE regime to be energetically favorable it is clear we must have

$$\mathcal{U}_{\text{QHE}}(B_{\text{ext}}) < \mathcal{U}_{\text{homog}}(B_{\text{ext}})$$

and this is equivalent to

$$\left( \frac{k^2 \xi^2 K_0(k\eta)}{2\pi} - 1 \right) \frac{(B_{\text{ext}} - B_n)^2}{8\pi} < \mathcal{U}_{\text{Peierls}}(B_{\text{ext}}). \quad (14)$$

As is easily seen, the QHE regime is always preferred whenever  $\xi < \frac{1}{k} \sqrt{2\pi/K_0(k\eta)}$ , which means that the Hall conductivity shows a perfect step-like dependence. However, in the limit of a very small  $\xi$  the impurities will strongly interact and the picture would no longer be applicable. In any case, this dirty limit should eventually destroy the QHE regime. When  $\xi > \frac{1}{k} \sqrt{2\pi/K_0(k\eta)}$  the inequality (14) predicts a plateau width

$$(\Delta B)_n = B_n \left\{ \frac{2n(n^2 + \alpha)}{(n^2 + \alpha)^2 - n^2} \right\}$$

where we have denoted  $\alpha = \frac{mac^2}{2e^2} \left( \frac{k^2 \xi^2 K_0(k\eta)}{2\pi} - 1 \right)$ . This result would clearly express the QHE stability if there are no alternative states of the system showing less energy.

The plateaus predicted by the above expressions, have been shown in Fig. 2 for four values of the quantity  $\alpha$ . We have used the same values for the physical parameters as those of the experiments described in [12].

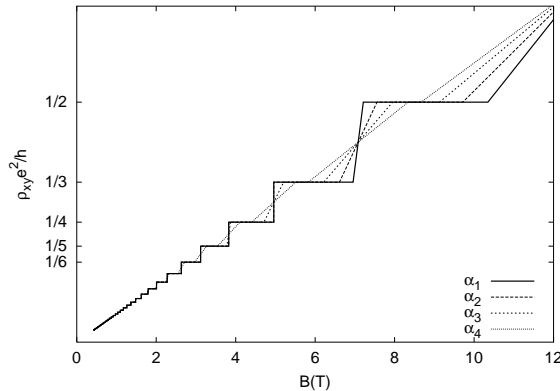


FIG. 2: Predicted plateaus for four values of the quantity  $\alpha$ .

Summarizing, it has been illustrated that the electromagnetic response of a superlattice of 2DEG free of impurities is able to transform an added distribution of impurities or defects



in a set of charge reservoirs. The excess charge over that required by an integral filling of Landau levels is then accumulated in the impurities. It can be understood from the present analysis, that it is the dynamically acquired capacity of accumulating charge of the impurities the main element to account for this effect. A next important step of the present study would be to investigate a similar model but in the case of planar samples. Analogous results can be expected in this case. The main additional complication seems to be the determination of the fields associated to a single defect, since that for that situation these fields spray out in the 3D space. An additional task for this planar case, would be the inclusion of the temperature. We expect this study to provide a foundation of the successful phenomenological model of Ingraham and Wilkes [15]. In addition, the present discussion can also be viewed as giving a microscopic explanation of the model of T. Toyoda et al [8]. These authors assumed the existence of a kind of particle reservoir being in equilibrium with the 2DEG and with a limited capacity to account for the plateau widths found in experiments. The fact that the total energy (11) becomes greater than the energy of the homogeneous magnetic field distribution can be expected to play the role of the limiting capacity stopping the particle accumulation and accounting for the plateau widths.

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